JOURNAL OF APPROXIMATION THEORY **35**, 64–76 (1982)

L_∞-Bound of L_2-Projections onto Splines on a Geometric Mesh*

Y. Y. FENG

Department of Mathematics, Chinese University of Science and Technology, Hofei, Anhui Province, People's Republic of China

AND

J. KOZAK

Matematika in Mehanika, Univerza Edvarda, Kardelja, Yugoslavia

Communicated by Carl de Boor

Received March 3, 1981

In [Feng and Kozak, *J. Approx. Theory* **32** (1981), 327–338], another proof of the boundedness of L_2-projections onto splines on a geometric mesh was given. In this paper, the sharp lower bound is obtained for the inverse of the corresponding B-spline Gram matrix. Namely, $\|G_r^{-1}\|_\infty = \Pi_{2k-1}(q^r; q)/\Pi_{2k-1}(-q^r; q)| \geqslant 2k-1$, for $r = k, k-1$.

1. INTRODUCTION

We begin with the explanation of some notations.

$$\Pi_n(\lambda; q) := \frac{1}{n!\,t^n} \sum_{i=0}^{n} (-)^{n-i} \binom{n}{i} \prod_{\substack{j=0 \\ j \neq i}}^{n} (q^j - \lambda)$$

is the generalized Euler–Frobenius polynomial of order n, and $t := \ln q$. $\binom{n}{r} := n!/r!(n-r)!$, is a binomial coefficient. $a_{n,i}(q)$ $(i = 0, 1,..., n-1)$ are the coefficients of the polynomial defined by

$$\sum_{i=0}^{n-1} a_{n,i}(q)\, \lambda^i := \frac{1}{\gamma_n (q-1)^n}\, \Pi_n(\lambda; q), \qquad \gamma_n := \frac{1}{n!\, t^n}.$$

* Sponsored by the United States Army under Contract No. DAAG29–80–C–0041.

0021-9045/82/050064–13$02.00/0

$a_{n,i}^{(j)}$ $(i=0, 1,..., n-1,\ \ j=0,..., i(n-1-i))$ are the coefficients of the polynomial defined by

$$a_{n,i}(q) =: q^{(n-i)(n-1-i)/2} \sum_{j=0}^{i(n-1-i)} a_{n,i}^{(j)} q^j.$$

Given a biinfinite geometric knot sequence $t := (q^i)_{-\infty}^{+\infty}$ for some $q \in (0, \infty)$ with

$$t_{\pm\infty} := \lim_{i\to\pm\infty} t_i, \qquad I := (t_{-\infty}, t_{+\infty}),$$

we denote by

$$N_{n,i} := (t_{i+n} - t_i)[t_i, t_{i+1},..., t_{i+n}](\cdot - x)_+^{n-1}$$

the corresponding B-splines normalized so that

$$\sum_i N_{n,i}(x) = 1$$

and by $S_{n,t} := \operatorname{span}\{N_{n,i}\}$ the space of splines of degree $n-1$ with knots t.

We can consider the projectors $P_{k,r}\colon C(I) \to S_{2k-r,t}$ defined by the conditions that

$$P_{k,r} f = \sum_i a_i(f)\, N_{2k-r,i}$$

and

$$\sum_j (N_{r,i}, N_{2k-r,j})\, a_j(f) = (N_{r,i}, f),$$

with $(f, g) := \int_a^b f(x)\, g(x)\, dx$. Then $P_{k,0}$ is the interpolation projector and $P_{k,k}$ the usual L_2-projector onto $S_{k,t}$.

This paper is a continuation of [1]. In [1] the uniform boundedness of

$$\|G_r^{-1}\|_\infty = \left|\frac{\Pi_{2k-1}(q^r; q)}{\Pi_{2k-1}(-q^r; q)}\right|$$

for $q \in (0, \infty)$ with $r = k,\ k-1$ was proved. Here, G_r^{-1} is the inverse of corresponding B-spline Gram matrix. In this paper we obtain the sharp lower bound for $\|G_r^{-1}\|_\infty$. Namely, we prove that for any $q \in (0, \infty)$ and $r = k$, $k-1$, the inequality

$$\|G_r^{-1}\|_\infty = \left|\frac{\Pi_{2k-1}(q^r; q)}{\Pi_{2k-1}(-q^r; q)}\right| \geqslant 2k-1$$

holds.

In order to prove this, we need some properties of $\Pi_n(\lambda; q)$ which were studied in [1] and [2]. For the reader's convenience we copy some of them as follows.

PROPOSITION 1 [2]. *$\Pi_n(\lambda; q)$ satisfies a "difference-delay" equation*

$$\Pi_0(\lambda; q) := 1,$$

$$\prod_{n+1}(\lambda; q) = \frac{1}{(n+1)\,t}\,((1-\lambda)\,q^n \Pi_n(q^{-1}\lambda; q)$$
$$-\,(q^{n+1}-\lambda)\,\Pi_n(\lambda; q)), \qquad n = 0, 1, \ldots.$$

PROPOSITION 2 [1]. *The polynomial $\Pi_n(\lambda; q)$ satisfies*

$$\Pi_n(\lambda; q) = \lambda^{n-1} q^{-n(n-1)/2} \Pi_n(q^n \lambda^{-1}; q). \tag{1.1}$$

The coefficients $a_{n,i}(q)$ can be computed recursively by

$$a_{n+1,i}(q) = (q-1)^{-1}\,((q^{n+1}-q^{n-i})\,a_{n,i}(q)$$
$$+\,(q^{n+1-i}-1)\,a_{n,i-1}(q)), \tag{1.2}$$

where

$$a_{n,0}(q) := 1,\ a_{n,-1}(q) = a_{n,n}(q) := 0.$$

PROPOSITION 3 [1]. *The coefficients $a_{n,i}(q)$ satisfy*

$$a_{n,i}(q) = q^{n(n-2i-1)/2} a_{n,n-1-i}(q), \tag{1.3}$$

and, for $n \geqslant 2$, the integer coefficients $a_{n,i}^{(j)}$ are symmetric

$$a_{n,i}^{(j)} = a_{n,i}^{(i(n-1-i)-j)}, \qquad \text{for all } j. \tag{1.4}$$

In particular

$$a_{n,i}^{(0)} = \binom{n-1}{i}, \tag{1.5}$$

$$a_{n,i}^{(1)} = (n-2)\binom{n-1}{i} - \binom{n-2}{i+1} - \binom{n-2}{i-2}. \tag{1.6}$$

2. THE SHARP LOWER BOUND FOR $\|G_r^{-1}\|_\infty$

Before proving the theorem we need to do some preparation.

LEMMA 2.1. *The following equalities hold*

$$\frac{\Pi_{2k-1}(-q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} = (-)^{k-1}\, q^{(3/2)k(k-1)} \sum_{j=0}^{(k-1)^2} (\alpha_{2k-1,j} - \beta_{2k-1,j})\, q^j$$

with

$$\begin{aligned} \alpha_{2k-1,j} &:= a^{(j)}_{2k-1,k-1} + \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-1,k-1-2i} + a^{(j-i(2i+1))}_{2k-1,k-1-2i}), \\ \beta_{2k-1,j} &:= a^{(j)}_{2k-1,k-2} + \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-1,k-2i} + a^{(j-i(2i+1))}_{2k-1,k-2-2i}). \end{aligned} \tag{2.1}$$

For convenience, here and below we use

$$a^{(r)}_{n,i} := 0, \qquad \text{if} \quad r < 0 \text{ or } r > i(n-1-i) \text{ as well as } i < 0. \tag{2.2}$$

In particular

$$\begin{aligned} \alpha_{2k-1,0} &= \binom{2k-2}{k-1}, \\ \beta_{2k-1,0} &= \binom{2k-2}{k-2}. \end{aligned} \tag{2.3}$$

Similarly

$$\frac{\Pi_{2k-2}(-q^{k-1};q)}{\gamma_{2k-2}(q-1)^{2k-2}} = (-)^{k-1}\, q^{(1/2)(k-1)(3k-4)} \sum_{j=0}^{(k-1)(k-2)} (\alpha_{2k-2,j} - \beta_{2k-2,j})\, q^j \tag{2.4}$$

with

$$\begin{aligned} \alpha_{2k-2,j} = \beta_{2k-2,j} &:= a^{(j)}_{2k-2,k-2} + \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-2,k-1-2i} + a^{(j-i(2i+1))}_{2k-2,k-2-2i}) \\ &= \sum_{i=0}^{\infty} a^{(j-(1/2)i(i+1))}_{2k-2,k-2-i}. \end{aligned}$$

Proof. By the definition of $a_{n,i}(q)$ and $a^{(i)}_{n,i}$,

$$\begin{aligned} \frac{\Pi_{2k-1}(-q^k;q)}{\gamma_{2k-1}(q-1)^{2k-1}} &= \sum_{i=0}^{2k-2} a_{2k-1,i}(q)(-q^k)^i \\ &= \sum_{i=0}^{2k-2} \sum_{j=0}^{i(2k-2-i)} (-)^i\, q^{\Phi_i+j} a^{(j)}_{2k-1,i} \end{aligned}$$

with

$$\Phi_i := \tfrac{1}{2}(2k-1-i)(2k-2-i) + ik,$$
$$\min_{0 \leqslant i \leqslant 2k-2} \Phi_i = q^{(3/2)k(k-1)}.$$

Let

$$\varphi_i := \Phi_i - \tfrac{3}{2}\, k(k-1) = \tfrac{1}{2}\,(k-1-i)(k-2-i).$$

Then

$$\begin{aligned}
&\frac{\Pi_{2k-1}(-q^k; q)}{\gamma_{2k-1}(q-1)^{2k-1}} \\
&\quad = q^{(3/2)k(k-1)} \sum_{i=0}^{2k-2} \sum_{j=0}^{i(2k-2-i)} (-)^i\, a^{(j)}_{2k-1,i} q^{\varphi_i + j} \\
&\quad = q^{(3/2)k(k-1)} \left[\sum_{i=0}^{k-1} \sum_{j=0}^{(k-1)^2 - i^2} (-)^{k-1-i}\, q^{(1/2)i(i-1)+j} a^{(j)}_{2k-1,k-1-i} \right. \\
&\qquad \left. + \sum_{i=1}^{k-1} \sum_{j=0}^{(k-1)^2 - i^2} (-)^{k-1+i}\, q^{(1/2)i(i+1)+j} a^{(j)}_{2k-1,k+i-1} \right] \\
&\quad = (-)^{k-1}\, q^{(3/2)k(k-1)} \left[\sum_{j=0}^{(k-1)^2} a^{(j)}_{2k-1,k-1} q^j + \sum_{i=1}^{k-1} \sum_{j=0}^{(k-1)^2 - i^2} \right. \\
&\qquad \left. \times (-)^i (q^{(1/2)i(i-1)+j} a^{(j)}_{2k-1,k-1-i} + q^{(1/2)i(i+1)+j} a^{(j)}_{2k-1,k-1-i}) \right] \\
&\quad = (-)^{k-1}\, q^{(3/2)k(k-1)} \sum_{j=0}^{(k-1)^2} \left[(a^{(j)}_{2k-1,k-1} - a^{(j)}_{2k-1,k-2}) \right. \\
&\qquad + \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-1,\, k-1-2i} + a^{(j-i(2i+1))}_{2k-1,k-1-2i}) \\
&\qquad \left. - \sum_{i=1}^{\infty} (a^{(j-i(2i-1))}_{2k-1,k-2i} + a^{(j-i(2i+1))}_{2k-1,k-2-2i}) \right] q^j,
\end{aligned}$$

and (2.1) follows.

By (2.1) and (1.5) we get

$$\alpha_{2k-1,0} = a^{(0)}_{2k-1,k-1} = \binom{2k-2}{k-1},$$

$$\beta_{2k-1,0} = a^{(0)}_{2k-1,k-2} = \binom{2k-2}{k-2}.$$

The same kind of argument proves (2.4).

An analogous argument gives

$$\frac{\Pi_{2k-1}(q^k; q)}{\gamma_{2k-1}(q-1)^{2k-1}} = q^{(3/2)k(k-1)} \sum_{j=0}^{(k-1)^2} (\alpha_{2k-1,j} + \beta_{2k-1,j})\, q^j, \tag{2.5}$$

$$\frac{\Pi_{2k-2}(q^{k-1}; q)}{\gamma_{2k-2}(q-1)^{2k-2}} = q^{(1/2)(k-1)(3k-4)} \sum_{j=0}^{(k-1)(k-2)} (\alpha_{2k-2,j} + \beta_{2k-2,j})\, q^j. \tag{2.6}$$

A straightforward calculation starting with the formula defining $\Pi_n(\lambda; q)$ leads to the expressions

$$\begin{aligned}
&\frac{\Pi_{2k-1}(q^k; q)}{\gamma_{2k-1}(q-1)^{2k-1}} \\
&\quad = \binom{2k-1}{k} q^{(3/2)k(k-1)} \prod_{i=1}^{k-2} (1 + q + \cdots + q^i)^2 (1 + q + \cdots + q^{k-1}) \\
&\quad =: \binom{2k-1}{k} q^{(3/2)k(k-1)} \sum_{j=0}^{(k-1)^2} d_{2k-1,j}\, q^j,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\frac{\Pi_{2k-2}(q^{k-1}, q)}{\gamma_{2k-2}(q-1)^{2k-2}} &= \binom{2k-2}{k-1} q^{(1/2)(k-1)(3k-4)} \prod_{i=1}^{k-2} (1 + q + \cdots + q^i)^2 \\
&=: \binom{2k-2}{k-1} q^{(1/2)(k-1)(3k-4)} \sum_{j=0}^{(k-1)(k-2)} d_{2k-2,j}\, q^j.
\end{aligned} \tag{2.8}$$

From (2.5), (2.6), (2.7), (2.8) it is easy to find the relations

$$\binom{2k-1}{k} d_{2k-1,i} = \alpha_{2k-1,i} + \beta_{2k-1,i}, \tag{2.9}$$

$$\binom{2k-2}{k-1} d_{2k-2,i} = \alpha_{2k-2,i} + \beta_{2k-2,i} = 2\alpha_{2k-2,i}, \tag{2.10}$$

$$d_{2k-1,i} = \sum_{j=0}^{k-1} d_{2k-2,i-j}, \tag{2.11}$$

$$d_{2k-2,i} = \sum_{j=0}^{k-2} d_{2k-3,i-j}, \tag{2.12}$$

and

$$d_{2k-1,i} = d_{2k-1,(k-1)^2-i}, \tag{2.13}$$

$$d_{2k-2,i} = d_{2k-2,(k-1)(k-2)-i}, \tag{2.14}$$

where

$$d_{r,i} := \alpha_{n,i} := 0 \qquad \text{if} \quad i < 0.$$

LEMMA 2.2. *The following equality holds*:

$$a_{n,i}^{(l)} = \sum_{r=l-i}^{l} a_{n-1,i}^{(r)} + \sum_{r=l+i-n+1}^{l} a_{n-1,i-1}^{(r)}. \tag{2.15}$$

Proof. By (1.2) and the definition of $a_{n,i}^{(i)}$

$$a_{n,i}(q) = q^{(n-i)(n-1-i)/2} \sum_{l=0}^{i(n-1-i)} a_{n,i}^{(l)} q^l \tag{$*$}$$

and

$$\begin{aligned} a_{n,i}(q) &= q^{n-1-i}(1+q+\cdots+q^i)\, a_{n-1,i}(q) \\ &\quad + (1+q+\cdots+q^{n-i-1})\, a_{n-1,i-1}(q) \qquad (2.16) \\ &= q^{(n-i)(n-1-i)/2} \sum_{j=0}^{i(n-1-i)} \left(\sum_{r=j-i}^{j} a_{n-1,i}^{(r)} + \sum_{r=j+i-n+1}^{j} a_{n-1,i-1}^{(r)} \right) q^j. \end{aligned}$$

Comparing with ($*$), (2.15) follows.

COROLLARY 2.1. *The following inequalities hold*:

$$a_{n,i}^{(l)} \geqslant a_{n,i}^{(l-1)} \qquad \text{for} \quad l \leqslant [\tfrac{1}{2}i(n-i-1)], \quad 0 \leqslant i \leqslant n-1 \tag{2.17}$$

and

$$a_{n,i}^{(l)} \geqslant a_{n,i-1}^{(l)} \qquad \text{for} \quad 0 \leqslant l \leqslant i(n-1-i), i \leqslant [(n-1)/2]. \tag{2.18}$$

Proof. We use mathematical induction to prove (2.17), (2.18). Suppose for $n-1$, (2.17), (2.18) hold. Using (2.15),

$$a_{n,i}^{(l)} - a_{n,i}^{(l-1)} = (a_{n-1,i}^{(l)} - a_{n-1,i}^{(l-i-1)}) + (a_{n-1,i-1}^{(l)} - a_{n-1,i-1}^{(l+i-n)}).$$

Since by (1.4),

$$a_{n,i}^{(l)} = a_{n,i}^{(i(n-1-i)-l)}$$

and

$$l \leqslant \tfrac{1}{2}i(n-1-i), \qquad 0 \leqslant i \leqslant n-1,$$

we get

$$a_{n-1,i}^{(l)} \geqslant a_{n-1,i}^{(l-i-1)}, \qquad \text{if} \quad l \leqslant \tfrac{1}{2}i(n-2-i),$$

$$a_{n-1,i}^{(l)} \geqslant a_{n-1,i}^{[(1/2)i(n-2-i)-i/2]} \geqslant a_{n-1,i}^{(l-i-1)}, \qquad \text{if} \quad \tfrac{1}{2}i(n-2-i) < l \leqslant \tfrac{1}{2}i(n-1-i).$$

However,

$$a_{n-1,i}^{(l)} \geqslant a_{n-1,i}^{(l-i-1)} \qquad \text{for} \quad 0 \leqslant l \leqslant \tfrac{1}{2}\, i(n-1-i).$$

The same kind of argument shows

$$a_{n-1,i-1}^{(l)} \geqslant a_{n-1,i-1}^{(l+i-n)}.$$

Now, we bring the induction hypothesis to the next level and (2.17) is proved since it obviously holds for $n=2$.

In order to prove (2.18), it is enough to prove 2.18), only for $0 \leqslant l \leqslant \frac{1}{2}(i-1)(n-i)$ because of (2.17) and (1.4). By (2.15)

$$a_{n,i}^{(l)} - a_{n,i-1}^{(l)} = \sum_{r=l-i+1}^{l} \left(a_{n-1,i}^{(r)} - a_{n-1,i-1}^{(r)}\right) + \sum_{r=l+i-n+1}^{l} \left(a_{n-1,i-1}^{(r)} - a_{n-1,i-2}^{(r)}\right) + \left(a_{n-1,i}^{(l-i)} - a_{n-1,i-2}^{(l+i-n)}\right).$$

By induction hypothesis and (2.17), we know

$$a_{n-1,i-1}^{(r)} \geqslant a_{n-1,i-2}^{(r)}$$

and

$$a_{n-1,i}^{(r)} - a_{n-1,i-1}^{(r)} \geqslant 0$$

as well as

$$a_{n-1,i}^{(l-i)} \geqslant a_{n-1,i-1}^{(l-i)} \geqslant a_{n-1,\, i-1}^{(l+i-n)} \geqslant a_{n-1,i-2}^{(l+i-n)}.$$

Therefore (2.18) holds for n and so (2.18) is proved, since it is obviously right for $n=2$.

LEMMA 2.3. *The following equalities hold*:

$$\begin{aligned} a_{2k-1,j} = 2 \sum_{r=j-k+1}^{j} a_{2k-2,k-1}^{(r)} + \sum_{i=1}^{\infty} \Bigg(& \sum_{r=j-k+1-i(2i-3)}^{j-i(2i-1)} a_{2k-2,k-1-2i}^{(r)} \\ & + \sum_{r=j-k+1-i(2i+1)}^{j-i(2i-1)} a_{2k-2,k-2-2i}^{(r)} + \sum_{r=j-k+1-i(2i-1)}^{j-i(2i+1)} \\ & \times a_{2k-2,k-1-2i}^{(r)} + \sum_{r=j-k+1-i(2i+3)}^{j-i(2i+1)} a_{2k-2,k-2-2i}^{(r)} \Bigg), \end{aligned} \tag{2.19}$$

$$\sum_{i=j-k+1}^{j} a_{2k-2,i} = \sum_{r=j-k+1}^{j} a^{(r)}_{2k-2,k-2} + \sum_{i=1}^{\infty} \left(\sum_{r=j-k+1-i(2i-1)}^{j-i(2i-1)} a^{(r)}_{2k-2,k-1-2i} + \sum_{r=j-k+1-i(2i+1)}^{j-i(2i+1)} a^{(r)}_{2k-2,k-2-2i} \right). \tag{2.20}$$

Proof. From (2.1), (2.4) and Lemma 2.2, by straightforward calculations, (2.19) and (2.20) can be proved. ∎

LEMMA 2.4. *For $j \leqslant [\frac{1}{2}(k-1)^2]$, the inequality*

$$a_{2k-1,j} \leqslant a_{2k-1,0} \cdot d_{2k-1,j}$$

holds.

Proof. Since

$$\begin{aligned} a_{2k-1,0} \cdot d_{2k-1,j} &= \binom{2k-2}{k-1} \sum_{l=j-k+1}^{j} d_{2k-2,l} \\ &= 2 \sum_{l=j-k+1}^{j} a_{2k-2,l}, \end{aligned} \tag{2.21}$$

in order to prove Lemma 2.4 it is enough to show

$$a_{2k-1,j} \leqslant 2 \sum_{l=j-k+1}^{j} a_{2k-2,l}. \tag{2.22}$$

From (2.19) and (2.20),

$$\begin{aligned} 2 \sum_{l=j-k+1}^{j} a_{2k-2,l} - a_{2k-1,j} = \sum_{i=1}^{\infty} \Bigg(&\sum_{r=j-k+1-i(2i-1)}^{j-k-i(2i-3)} a^{(r)}_{2k-2,\, k-1-2i} + \sum_{r=j+1-i(2i+1)}^{j-i(2i-1)} \\ &\times a^{(r)}_{2k-2,k-1-2i} - \sum_{r=j+1-i(2i+1)}^{j-i(2i-1)} a^{(r)}_{2k-2,k-2-2i} \\ &- \sum_{r=j-k+1-i(2i+3)}^{j-k-i(2i+1)} a^{(r)}_{2k-2,\, k-2-2i} \Bigg) \end{aligned}$$

$$= \sum_{i=1}^{\infty} \left(\sum_{r=j+1-i(2i+1)}^{j-i(2i-1)} (a^{(r)}_{2k-2,k-1-2i} - a^{(r)}_{2k-2,k-2-2i}) \right.$$
$$\left. + \sum_{r=j-k+1-i(2i+1)}^{j-k-i(2i-3)} (a^{(r)}_{2k-2,k-1-2i} - a^{(r-4i)}_{2k-2,k-2-2i}) \right).$$

Because of (2.17), (2.18) and (1.4), Lemma 2.4 follows. ∎

THEOREM. *For any $q \in (0, \infty)$ and for $r = k-1, k$, the inequality*

$$\|G_r^{-1}\| = \left| \frac{\Pi_{2k-1}(q^r; q)}{\Pi_{2k-1}(-q^r; q)} \right| \geqslant 2k-1 \tag{3.0}$$

holds, and $\lim_{q\to\infty} \|G_r^{-1}\| = 2k-1$.

Proof. Because of the symmetry of $\Pi_n(\lambda; q)$, we can restrict our discussion to the case $q \in [1, \infty)$.

By (2.1), (2.3), (2.7), Lemma 2.4 and (1.1), we have

$$\left| \frac{\Pi_{2k-1}(q^{k-1}; q)}{\Pi_{2k-1}(-q^{k-1}; q)} \right| = \left| \frac{\Pi_{2k-1}(q^k; q)}{\Pi_{2k-1}(-q^k; q)} \right|$$
$$= \frac{\binom{2k-1}{k} \sum_{i=0}^{(k-1)^2} d_{2k-1,i} q^i}{\sum_{i=0}^{(k-1)^2} (2\alpha_{2k-1,i} - \binom{2k-1}{k} d_{2k-1,i}) q^i}$$
$$\geqslant \frac{\binom{2k-1}{k} \sum_{i=0}^{(k-1)^2} d_{2k-1,i} q^i}{\sum_{i=0}^{(k-1)^2} (2\alpha_{2k-1,0} \cdot d_{2k-1,i} - \binom{2k-1}{k} d_{2k-1,i}) q^i}$$
$$= 2k-1,$$

and we refer to [1] for equality.

The proof of the theorem relies mainly on Lemma 2.4. In order to prove the monotonicity of

$$\left| \frac{\Pi_{2k-1}(q^k; q)}{\Pi_{2k-1}(-q^k; q)} \right| \qquad \text{for} \quad q \in (0, \infty),$$

it is sufficient to prove the stronger inequality

$$\frac{\alpha_{2k-1,j}}{\alpha_{2k-1,j+1}} \geqslant \frac{d_{2k-1,j}}{d_{2k-1,j+1}} \qquad \text{for} \quad 0 \leqslant j \leqslant \left[\frac{1}{2} (k-1)^2 \right].$$

We have failed to prove this inequality. But numerical results (see tables) show the inequality is true at least for $n \leqslant 9$.

3. APPENDIX

$\alpha_{n,i}, \beta_{n,i}, d_{n,i}$ for $4 \leqslant n \leqslant 9$, $i = 0, 1, \ldots, [(n-1)/2] \cdot [n/2]$

n	α, β, d	i 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	$\alpha = \beta$	3	6	3														
	d	1	2	1														
5	α	6	17	22	17	6												
	β	4	13	18	13	4												
	d	1	3	4	3	1												
6	$\alpha = \beta$	10	40	80	100	80	40	10										
	d	1	4	8	10	8	4	1										
7	α	20	96	242	422	548	548	422	242	96	20							
	β	15	79	213	383	502	502	383	213	79	15							
	d	1	5	13	23	30	30	23	13	5	1							
8	$\alpha = \beta$	35	210	665	1470	2485	3360	3710	3360	2485	1470	665	210	35				
	d	1	6	19	42	71	96	106	96	71	42	19	6	1				
9	α	70	476	1728	4449	9005	15073	21448	26354	28202	26354	21448	15073	9005	4449	1728	476	70
	β	56	406	1548	4119	8509	14411	20636	25432	27238	25432	20636	14411	8509	4119	1548	406	56
	d	1	7	26	68	139	234	334	411	440	411	334	234	139	68	26	7	1

$a_{n,i}^{(j)}\, 4 \leqslant n \leqslant 9,\ i = 0, 1, [(n-1)/2],\ j = 0, 1, \ldots, i(n-1-i)$

n	I	J 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	0	1																
	1	3	5	3														
5	0	1																
	1	4	9	9	4													
	2	6	16	22	16	6												
6	0	1																
	1	5	14	19	14	5												
	2	10	35	66	80	66	35	10										
7	0	1																
	1	6	20	34	34	20	6											
	2	15	64	149	233	269	233	149	64	15								
	3	20	90	222	382	494	494	382	222	90	20							
8	0	1																
	1	7	27	55	69	55	27	7										
	2	21	105	288	540	765	855	765	540	288	105	21						
	3	35	189	560	1175	1918	2540	2785	2540	1918	1175	560	189	35				
9	0	1																
	1	8	35	83	125	125	83	35	8									
	2	28	160	503	1091	1806	2400	2632	2400	1806	1091	503	160	28				
	3	56	350	1198	2913	5561	8767	11736	13536	13536	11736	8767	5561	2913	1198	350	56	
	4	70	448	1568	3918	7754	12764	17956	21916	23402	21916	17956	12764	7754	3918	1568	448	70

ACKNOWLEDGMENTS

We would like to thank Professor Carl de Boor for his valuable help.

REFERENCES

1. Y. Y. FENG AND J. KOZAK, On the generalized Euler–Frobenius polynomial, *J. Approx. Theory* **32** (1981), 327–338.
2. A. MICCHELLI, Cardinal L-splines, *in* "Studies in Spline Functions and Approximation Theory," pp. 203–250, Academic Press, New York/London, 1976.